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Static fluid cylinders in general relativity

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Abstract. Einstein's field equations are applied to infinite cylinders of fluid, and some exact solutions are obtained by a systematic method. Solutions having $p \propto \rho$ are shown to be unstable as the equation of state $p = \rho$ is approached.

1. Introduction

From the earliest days of general relativity, there has been continuing interest in cylindrically symmetric solutions, which offer the simplest means of exploring departures from spherical symmetry. The static vacuum solution dates back to the work of Levi-Civita and Weyl in 1917–18, and its extension to rotating sources has intermittently occupied other researchers almost to the present day (see Lewis 1932, Frehland 1972, and references therein).

However, the discovery of explicit source solutions has generally lagged behind the vacuum fields, even in the supposedly simple case of a static distribution of fluid. The solutions of Marder (1958), for example, have a specialized metric and anisotropic pressure. There has in fact been as much success with rotating dust solutions (King 1974, Zimmerman 1975) as with any other type, despite the complications of angular momentum.

The purpose of this paper is to present methods of generating static cylindrical perfect-fluid solutions, and to give some simple examples.

2. Metric and field equations

We choose the static cylindrical metric to be

$$ds^2 = e^{2\nu}\, dt^2 - dr^2 - e^{2\lambda}\, d\phi^2 - e^{2\mu}\, dz^2 \tag{2.1}$$

where λ, μ, ν are functions of the radial coordinate, r. In vacuum solutions for this metric, two functions will suffice (cf Synge 1960, chap. 8, § 1). The solutions of Marder (1958) have only two independent functions, and are therefore restricted from the outset.

Using relativistic units ($c = G = 1$), the energy tensor for a perfect fluid in this situation is

$$T^{ij} = \mathrm{diag}(p, p e^{-2\lambda}, p e^{-2\mu}, \rho e^{-2\nu}) \tag{2.2}$$

where $(x^1, x^2, x^3, x^4) \equiv (r, \phi, z, t)$. With d/d$r$ denoted by a prime, the field equations $G^{ij} = -8\pi T^{ij}$ give

$$\lambda'\mu' + \mu'\nu' + \nu'\lambda' = 8\pi p \tag{2.3}$$

$$\mu'' + \nu'' + \mu'^2 + \mu'\nu' + \nu'^2 = 8\pi p \tag{2.4}$$

$$\lambda'' + \nu'' + \lambda'^2 + \lambda'\nu' + \nu'^2 = 8\pi p \tag{2.5}$$

$$\lambda'' + \mu'' + \lambda'^2 + \lambda'\mu' + \mu'^2 = -8\pi\rho. \tag{2.6}$$

The conservation equation $T^{1j}{}_{;j} = 0$ yields the condition for static equilibrium

$$(\rho + p)\nu' + p' = 0. \tag{2.7}$$

Subtracting (2.4) from (2.5), we can integrate to obtain

$$(\lambda - \mu)' \exp(\lambda + \mu + \nu) = 1 \tag{2.8}$$

where the constant on the right is determined by imposing realistic conditions at the axis: as $r \downarrow 0$, we want $e^\lambda/r \to 1$, and $\mu, \mu', \nu \to 0$. Also, the equation (2.4)+(2.5)−2×(2.3) is

$$(\lambda + \mu + 2\nu)'' + 2\nu'^2 + (\lambda - \mu)'^2 - \nu'(\lambda + \mu)' = 0. \tag{2.9}$$

If we define

$$\eta = \lambda - \mu$$
$$\zeta = \lambda + \mu + \nu \tag{2.10}$$

then (2.8) becomes

$$\eta' = e^{-\zeta} \tag{2.11}$$

and consequently (2.9) may be written as

$$\nu'' + 3\nu'^2 - \zeta'\nu' + \zeta'' + e^{-2\zeta} = 0. \tag{2.12}$$

From this we get a second-order linear equation for $u = e^{3\nu}$:

$$u'' - \zeta' u' + 3(\zeta'' + e^{-2\zeta})u = 0. \tag{2.13}$$

If no equation of state is specified, then ζ may be assigned arbitrarily, though we note that a physical solution must have $e^\zeta/r \to 1$ as $r \downarrow 0$.

The entire solution is reducible to quadratures if a particular integral of (2.13), say $u = u^*$, can be found. For then the substitution $u = u^* v$ gives

$$v'' + (2 \ln u^* - \zeta)' v' = 0 \tag{2.14}$$

and hence the general integral of (2.13) is

$$u = au^* + bu^* \int (e^\zeta / u^{*2}) \, dr \tag{2.15}$$

where a and b are the constants of integration. We can set $u(0) = 1$, so that the metric is Minkowskian at the axis: then only one of the two constants is arbitrary. (We get $u'(0) = 0$ automatically if a suitable ζ is chosen.) Finally, (2.11) provides a quadrature for η, with the integration constant being determined by the requirement that $e^\eta/r \to 1$ as $r \downarrow 0$. The whole process of integration thus gives rise to only one arbitrary constant of physical significance, though others may be incorporated into ζ.

Having obtained the metric functions, one has to work out p and ρ to check that a physically plausible solution is possible. In terms of ζ and ν, we find from (2.3)

$$32\pi p = (\zeta' + 3\nu')(\zeta' - \nu') - e^{-2\zeta} \tag{2.16}$$

and then ρ is most easily found from

$$4\pi(5p - \rho) = (e^{\zeta})''/e^{\zeta} \tag{2.17}$$

which is equivalent to $3\times(2.3)+(2.4)+(2.5)+(2.6)$. If we insist that $\rho > 0, p \geqslant 0$, then from (2.7) the condition $p' \leqslant 0$ corresponds to $\nu' \geqslant 0$, or equivalently to $u' \geqslant 0$. (There is equality only at $r = 0$.) A realistic equation of state will also give $\rho' < 0$ when $p' < 0$, and in two of the examples below this property is readily verified using (2.17).

3. Some examples

3.1. *Model* (*i*)

An especially simple solution that is also physically interesting is obtained by setting $e^{\zeta} = r$. In this case, (2.13) immediately gives $u = 1 + \beta^2 r^2$, where β is a constant. The complete solution is

$$\begin{aligned} e^{2\lambda}/r^2 = e^{2\mu} &= (1+x^2)^{-1/3} \\ e^{2\nu} &= (1+x^2)^{2/3} \\ 40\pi p = 8\pi\rho &= (5\beta^2/3)(1+x^2)^{-2} \end{aligned} \tag{3.1}$$

where $x = \beta r$. We thus have the highly relativistic equation of state $p = \rho/5$.

3.2. *Model* (*ii*)

The simplicity of model (i) is due to the fact that $e^{\zeta} = r$ gives $\zeta'' + e^{-2\zeta} = 0$ in (2.13). There are two other cases having this property: $e^{\zeta} = (1/\beta)\sin(\beta r)$, and $e^{\zeta} = (1/\beta)\sinh(\beta r)$, where β is a non-zero constant.

The first of these is physically acceptable. We find

$$u = e^{3\nu} = 1 + \alpha \sin^2(\tfrac{1}{2}\beta r) \tag{3.2}$$

where α is the constant of integration. This leads to

$$96\pi p = \beta^2\left(\frac{\alpha+1}{u^2} - 4\right) \tag{3.3}$$

and hence $p(0) > 0$ requires $\alpha > 3$. At the boundary ($p = 0$), we have

$$\sin^2(\tfrac{1}{2}\beta r) = [\sqrt{(1+\alpha)} - 2]/2\alpha. \tag{3.4}$$

For any fixed β, the radius of the boundary is greatest when $\alpha = 6 + 4\sqrt{3}$, giving $\beta r = \pi/6$. Equation (3.3) shows that $p' < 0$ when $0 < \beta r \leqslant \pi/6$. Finally, (2.17) implies that $\rho > 0$ when $p \geqslant 0$, and that $\rho' = 5p'$.

The second case is obtained from the first on replacing β by $i\beta$, and (2.17) shows at once that it is unphysical: $\rho < 0$ when $p = 0$. Similar properties are found in (i) above, and in (iii), (iv) below: these models are acceptable as they stand, but become unphysical if we replace β by $i\beta$.

3.3. Model (iii)

Setting $e^{\zeta} = (1/\beta)\tanh(\beta r)$ leads to

$$u = \cos L + \alpha \sin L$$

where (3.5)

$$L = \sqrt{3} \ln \cosh(\beta r).$$

Putting $\beta r = x$, we obtain

$$32\pi p = \beta^2\left[\operatorname{sech}^2 x\left(\frac{2}{\sqrt{3}} \tan\theta - \sinh^2 x \tan^2\theta - 1\right) - 1\right]$$

where (3.6)

$$\theta = \tan^{-1}\alpha - L.$$

We require $\alpha > \sqrt{3}$, to give $p(0) > 0$. Since $p < 0$ when $\tan\theta = 0$, the distribution has $L < \tan^{-1}\alpha < \pi/2$, giving $x < \cosh^{-1}\exp(\pi/2\sqrt{3}) \simeq 1{\cdot}56$, for all values of α. (Computer-generated models indicate that $x \leqslant 0{\cdot}38$ at the boundary, the maximum occurring when $\alpha \simeq 7{\cdot}2$.)

From (2.17) we find that $\rho > 0$ when $p \geqslant 0$, and (2.7) then implies that $p' < 0$ between the axis and the boundary. Differentiating (2.17), we have $\rho' < 0$ when $p' < 0$.

3.4. Model (iv)

If we choose $e^{\zeta} = r(1 - \beta^2 r^2)$, then

$$u = \alpha(1 - \beta^2 r^2)^q + (1 - \alpha)(1 - \beta^2 r^2)^{2-q}$$

where (3.7)

$$q = 1 + \frac{\sqrt{13}}{2} \simeq 2{\cdot}8.$$

The expression for p is rather unwieldy, and we note only that $p(0) > 0$ requires $\alpha < (\sqrt{13} - 11)/2\sqrt{13} \simeq -1$. From (2.17) we get $\rho > 5p$, and then (2.7) shows that $p' < 0$ between the axis and the boundary. In this example we cannot conclude from (2.17) that $\rho' < 0$ when $p' < 0$, but computer-generated solutions indicate that the property always holds. They also show that the greatest value of βr at the boundary is approximately $0{\cdot}21$, attained when $\alpha \simeq -5{\cdot}2$.

4. Fitting exterior solutions

We shall deal mainly with interior solutions filling all space, but in models (ii)–(iv) of § 3 the interior solution terminates at a finite radius, when the pressure becomes zero with the density still non-negative. It is therefore desirable to know how an exterior vacuum solution may be fitted smoothly to a given interior solution.

It is natural to choose (2.1) as the form of the exterior metric, and solve the field equations of § 2 with $p = \rho = 0$. We note first that (2.17) gives

$$\exp(\lambda + \mu + \nu) = k(r - a) \tag{4.1}$$

where a and k are constants. In the same way as (2.8) was derived earlier, we can now obtain with the aid of (4.1)

$$\begin{aligned} \mu' &= \lambda' + b/(r-a) \\ \nu' &= \lambda' + c/(r-a) \end{aligned} \tag{4.2}$$

where b and c are constants. Integrating (4.2), and substituting the results back into (4.1), we get

$$\begin{aligned} e^{\lambda} &= A(r-a)^{(1-b-c)/3} \\ e^{\mu} &= B(r-a)^{(1+2b-c)/3} \\ e^{\nu} &= C(r-a)^{(1-b+2c)/3} \end{aligned} \tag{4.3}$$

where A, B, C are further constants of integration. Substitution of (4.3) into (2.3) gives the condition

$$b^2 - bc + c^2 = 1 \tag{4.4}$$

and then the other field equations are automatically satisfied.

In order to fit an exterior solution to a given interior solution, at a boundary $r = s$, we use (4.3) to write

$$\begin{aligned} \lambda' &= \tfrac{1}{3}(1-b-c)/(r-a) \\ \mu' &= \tfrac{1}{3}(1+2b-c)/(r-a) \\ \nu' &= \tfrac{1}{3}(1-b+2c)/(r-a). \end{aligned} \tag{4.5}$$

In these equations we can put $r = s$, set λ', μ', ν' to their required (interior) boundary values, and solve for a, b, c. The condition for the existence of a solution turns out to be $\zeta' \neq 0$, which is guaranteed at the boundary by (2.16). The solution automatically satisfies (4.4), because the prescribed values of λ', μ' and ν' must give $p = 0$ in (2.3).

It is obvious from (4.3) that we can always choose A, B, C so that λ, μ, ν are continuous at the boundary. Also, if the interior solution has $\rho(s) = 0$ (in addition to $p(s) = 0$), then (2.4)–(2.6) imply that λ'', μ'', ν'' are continuous at the boundary as well.

5. Models with $p \propto \rho$

The result (2.17), and the simple solution (3.1), suggest that further integration of the equations might be possible if one took the equation of state $\rho = np$, where n is a constant. (This is more convenient than the usual form $p = (\gamma - 1)\rho$.) Substituting for ρ in (2.17), and then eliminating p by using (2.16), we obtain

$$\left(\frac{8}{5-n}\right)(\zeta'' + \zeta'^2) = \zeta'^2 + 2\zeta'\nu' - 3\nu'^2 - e^{-2\zeta} \tag{5.1}$$

provided $n \neq 5$. (We covered $n = 5$ in § 3.1.) Subtracting (2.12) from (5.1) gives

$$k(\zeta'' + \zeta'^2) = \nu'' + \zeta'\nu' \tag{5.2}$$

where $k = (3+n)/(5-n)$, and this has the integral

$$\nu' = k(\zeta' - e^{-\zeta}). \tag{5.3}$$

With $y = e^\zeta$, elimination of ν' from (2.12) by means of (5.3) now yields

$$yy'' + \alpha y'^2 - 2(\alpha+1)y' + (\alpha+2) = 0 \tag{5.4}$$

where $\alpha = \frac{1}{2}(n+7)(n-1)/(5-n)$. This equation may be recast into the form

$$\frac{y'\,dy'}{(y'-1)[\alpha y' - (\alpha+2)]} + \frac{dy}{y} = 0. \tag{5.5}$$

If $\alpha \neq 0$, we get the integral

$$\frac{y^2|1-y'/a|^a}{1-y'} = A \tag{5.6}$$

where

$$a = 1 + 2/\alpha = [(n+1)^2 + 12]/[(n+7)(n-1)]. \tag{5.7}$$

If $\alpha = 0$ ($n = 1$ or -7), the corresponding integral is

$$\frac{y^2\,e^{-y'}}{1-y'} = B \tag{5.8}$$

a result which follows directly from (5.5), or from (5.6) on letting $a \to \infty$.

To identify the integration constants A and B in terms of the equation of state and the central pressure or density, consider the behaviour of y near the axis. For small r, we want

$$y \approx r(1+\beta r^2), \qquad \beta = \text{constant}. \tag{5.9}$$

Using (5.3), we then have from (2.16)

$$2\pi p(0) = 3\beta/(5-n). \tag{5.10}$$

Substituting (5.9) into (5.6), and eliminating β by means of (5.10), we find

$$A = \frac{|1-1/a|^a}{2\pi p(0)(n-5)}. \tag{5.11}$$

The corresponding result for (5.8) may be obtained by letting $a \to \infty$. For the case $n = 1$,

$$B = \frac{-1}{8\pi\, e p(0)}. \tag{5.12}$$

In the remainder of this section, we shall only consider $n > 1$.

Further integration of (5.6) or (5.8) must be carried out numerically, but asymptotic solutions as $r \to \infty$ can be deduced directly from these equations. In the case of (5.6), the assumption that y' diverges, so that $y^2(y')^{a-1} \approx$ constant, leads to its own contradiction. Therefore $y' \to$ constant, and clearly the only possibility is $y' \to a \neq 1$. (Since $\alpha \neq -2$, $\forall n$, (5.4) forbids $y \to \sqrt{A}$.) Suppose then that

$$y' \approx a + br^m, \qquad m < 0 \tag{5.13}$$

giving $y \approx ar$, to sufficient accuracy. Substituting into (5.6), we find two conditions:

$$|b|^a = A|a|^{a-2}(1-a) \tag{5.14}$$

and

$$am + 2 = 0 \tag{5.15}$$

the second of which gives

$$m = \frac{-2(n+7)(n-1)}{(n+1)^2+12}. \tag{5.16}$$

Thus for $n > 1$ we have $m < 0$, as is required in (5.13). The sign of b, though not given by (5.14), is determined by the fact that $y' \in [1, a)$ or $(a, 1]$, according as $a > 1$ or $a < 1$. (This can be seen from (5.6).) Thus from (5.13), b has the same sign as $1-a$.

The definitions (2.10), with (2.11) and (5.3), give

$$\begin{aligned} \lambda' &= \frac{(n-1)y'-4}{(n-5)y} \\ \mu' &= \left(\frac{n-1}{n-5}\right)\left(\frac{y'-1}{y}\right) \\ \nu' &= \left(\frac{n+3}{n-5}\right)\left(\frac{1-y'}{y}\right). \end{aligned} \tag{5.17}$$

Hence we have asymptotic power-of-r behaviour in e^{λ}, e^{μ} and e^{ν}, with the power law indices determined by

$$\begin{aligned} \lambda' &\approx \frac{1}{r}\frac{(n-1)(n+3)}{(n+1)^2+12} \\ \mu' &\approx \frac{1}{r}\frac{-4(n-1)}{(n+1)^2+12} \\ \nu' &\approx \frac{1}{r}\frac{4(n+3)}{(n+1)^2+12} \end{aligned} \tag{5.18}$$

for $n > 1$. (See (5.7), (5.13).) Although we had to exclude $n = 5$ in order to obtain these results, the solution (3.1) fits this pattern. For other values of n, computer-generated solutions readily verify (5.18).

We note finally that (2.7) implies an asymptotic power law diminution of pressure and density. On integration it gives

$$\rho = np = \rho(0)\exp[-(n+1)\nu]. \tag{5.19}$$

6. The case $p = \rho$

Although one can pursue this special case further by means of (5.8) and (5.12), it is more convenient to use the original field equations. We note first that with $n = 1$ the second of equations (5.17) gives $\mu' = 0$, so that we can set $\mu = 0$ and obtain three independent equations from (2.3)–(2.6):

$$\lambda'\nu' = 8\pi\rho \tag{6.1}$$

$$\nu'' + \nu'^2 = 8\pi\rho \tag{6.2}$$

$$\lambda'' + \lambda'^2 = -8\pi\rho. \tag{6.3}$$

The conservation equation (2.7) is immediately integrable:

$$\rho = \rho(0)\, e^{-2\nu} \tag{6.4}$$

and substitution into (6.2) leads to a further integral:

$$\nu' = (2k)^{1/2} \nu^{1/2} \, e^{-\nu} \tag{6.5}$$

where $k = 8\pi p(0)$.

Substituting (6.4) and (6.5) into (6.1), we get

$$\lambda' = (\tfrac{1}{2}k)^{1/2} \nu^{-1/2} \, e^{-\nu} \tag{6.6}$$

and hence (using (6.5) again)

$$\lambda'' = -k \, e^{-2\nu}(1 + \tfrac{1}{2}\nu^{-1}). \tag{6.7}$$

Equation (6.3) is now automatically satisfied, while from (6.5) and (6.6) we have $\lambda' = \frac{1}{2}\nu'/\nu$, giving the integral

$$e^{2\lambda} = C\nu, \qquad C = \text{constant}. \tag{6.8}$$

To identify C, we use (6.8) and (6.5):

$$(e^{\lambda})' = (\tfrac{1}{2}Ck)^{1/2} \, e^{-\nu}. \tag{6.9}$$

At $r = 0$, we want $(e^{\lambda})' = e^{\nu} = 1$, and so $C = 2/k$.

We have now reduced the problem to that of integrating (6.5), which can be done numerically. This is an improvement on the method of § 5, which required the integration of (2.11) as well as (5.8).

For comparison with (5.18), we note that the asymptotic solution for large r has (from (6.5) and (6.8))

$$\begin{aligned} e^{2\nu} &\approx 2kr^2 \ln r \\ e^{2\lambda} &\approx \frac{2}{k} \ln r. \end{aligned} \tag{6.10}$$

Because of the logarithmic terms, this does not fit the pattern found for the neighbouring asymptotic solutions with $n > 1$.

7. Discussion

The methods and results outlined above go some way towards filling a gap in the literature on cylindrical metrics. One notable feature of the solutions is that they are unstable with respect to the equation of state: models having $\rho = np$, $n > 1$, do not join smoothly on to the 'stiff matter' case $\rho = p$. It is interesting that equation-of-state instability has also been found by Collins (1974) in the behaviour of homogeneous world models. Symmetry plays a central role in homogeneous universes, just as it does in the present static models, and it seems likely that the instability phenomenon is restricted to situations such as these.

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